TUTORIAL NOTES FOR MATH4220

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1. BOUNDARY VALUE PROBLEMS AND SEPARATION OF VARIABLES

Let us illustrate the separation of variables by studying some boundary value problems.

Example 1. Solve the following Dirichlet problem for Poisson equation over a sector in $\mathbb{R}^2$,

$$\begin{aligned} \rho^2\partial_\rho^2 u + \rho\partial_\rho u + \partial_\varphi^2 u = 0, &\quad 0 \le \rho < r, 0 < \varphi < \theta, \\ u(\rho,0) = u(\rho,\theta) = 0, &\quad 0 \le \rho \le r, \\ u(r,\varphi) = \sin\left(\frac{\pi}{\theta}\varphi\right), &\quad 0 \le \varphi \le \theta. \end{aligned}$$

Proof. Setting $u(\rho,\varphi) = R(\rho)\Phi(\varphi)$, then R and Φ satisfy

$$\rho^2\Phi R'' + \rho\Phi R' + R\Phi'' = 0.$$

We denote λ such that

$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

For Φ, we have

$$\Phi'' + \lambda\Phi = 0,$$

solving the second order differential equation. If $\lambda < 0$, then the general solution is

$$\Phi(\varphi) = a_1 e^{\sqrt{-\lambda}\varphi} + a_2 e^{-\sqrt{-\lambda}\varphi},$$

by the boundary condition at $\varphi = 0$ and $\varphi = \theta$, we have

$$a_1 = a_2 = 0,$$

therefore

$$\Phi(\varphi) = 0.$$

If $\lambda = 0$, then the general solution is

$$\Phi(\varphi) = a_1 + a_2\varphi,$$

by the boundary condition at $\varphi = 0$ and $\varphi = \theta$, we have

$$a_1 = a_2 = 0,$$

therefore

$$\Phi(\varphi) = 0.$$

If $\lambda > 0$, then the general solution is

$$\Phi(\varphi) = a_1\cos(\sqrt{\lambda}\varphi) + a_2\sin(\sqrt{\lambda}\varphi),$$

by the boundary condition at $\varphi = 0$ and $\varphi = \theta$, we have

$$\Phi(0) = \Phi(\theta) = 0,$$

then

$$a_1 = 0, \quad \lambda_n = \left(\frac{n\pi}{\theta}\right)^2,$$

for all $n \in \mathbb{N}$. Therefore

$$\Phi_n(\varphi) = a_n \sin(\frac{n\pi}{\theta}\varphi),$$

for all $n \in \mathbb{N}$, where a_n is to be determined.

For R, we have

$$\rho^2 R'' + \rho R' - \lambda R = 0,$$

Solving the second order differential equation, if $\lambda = 0$, then the general solution is

$$R(\rho) = b_1 + b_2 \ln \rho,$$

if $\lambda \neq 0$, then the general solution is

$$R(\rho) = b\rho^\alpha,$$

where α satisfies

$$\alpha(\alpha - 1) + \alpha - \lambda = 0.$$

By the above discussion for Φ, and note that u is bounded at the origin, we have

$$R_n(\rho) = b_n \rho^{\frac{n\pi}{\theta}},$$

for all $n \in \mathbb{N}$ where b_n is to be determined.

Therefore we seek the solution in the following form

$$u(\rho, \varphi) = \sum_{n=1}^{\infty} c_n \rho^{\frac{n\pi}{\theta}} \sin\left(\frac{n\pi}{\theta}\varphi\right).$$

By the boundary condition at $\rho = r$, we have

$$c_n = \frac{2}{\theta r^{\frac{n\pi}{\theta}}} \int_0^\theta \sin\left(\frac{\pi}{\theta}\varphi\right) \sin\left(\frac{n\pi}{\theta}\varphi\right) d\varphi = \begin{cases} r^{-\frac{\pi}{\theta}}, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

then the solution is

$$u(\rho, \varphi) = \left(\frac{\rho}{r}\right)^{\frac{\pi}{\theta}} \sin\left(\frac{\pi}{\theta}\varphi\right).$$

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Example 2. Solve the following Dirichlet problem for heat equation over an interval,

$$\begin{aligned} \partial_t u - \partial_x^2 u = 0, &\quad t > 0, 0 < x < l, \\ u(t, 0) = u(t, l) = 0, &\quad t > 0, \\ u(0, x) = \sin\left(\frac{\pi}{l}x\right), &\quad 0 < x < l. \end{aligned}$$

Solution. Setting $u(t, x) = T(t)X(t)$, then T and R satisfy

$$XT' - TX'' = 0.$$

We denote λ such that

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda.$$

For X, we have

$$X'' + \lambda X = 0,$$

solving the second order differential equation. If $\lambda < 0$, then the general solution is

$$X(x) = a_1 e^{\sqrt{-\lambda}x} + a_2 e^{-\sqrt{-\lambda}x},$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = a_2 = 0,$$

therefore

$$X(x) = 0.$$

If $\lambda = 0$, then the general solution is

$$X(x) = a_1 + a_2 x,$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = a_2 = 0,$$

therefore

$$X(x) = 0.$$

If $\lambda < 0$, then the general solution is

$$X(x) = a_1 \cos(\sqrt{\lambda} x) + a_2 \sin(\sqrt{\lambda} x),$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = 0, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2,$$

for all $n \in \mathbb{N}$. Therefore

$$X_n(x) = a_n \sin\left(\frac{n\pi}{l} x\right),$$

for all $n \in \mathbb{N}$, where a_n is to be determined.

For T, we have

$$T' + \lambda T = 0,$$

solving the first order differential equation, the general solution is

$$T(t) = b e^{-\lambda t}.$$

By the above discussion of X, we have

$$T_n(t) = b_n e^{-\left(\frac{n\pi}{l}\right)^2 t},$$

for all $n \in \mathbb{N}$ where b_n is to be determined.

Therefore we seek the solution in the following form

$$u(t,x) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n\pi}{l} t\right)^2} \sin\left(\frac{n\pi}{l} x\right).$$

By the initial condition, we have

$$c_n = \frac{2}{l} \int_0^l \sin\left(\frac{\pi}{l} x\right) \sin\left(\frac{n\pi}{l} x\right) dx = \begin{cases} 1, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

then the solution is

$$u(t,x) = e^{-\left(\frac{\pi}{l} t\right)^2} \sin\left(\frac{\pi}{l} x\right).$$

Example 3. Solve the following Dirichlet problem for wave equation over an interval,

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u = 0, &\quad t > 0, 0 < x < l, \\ u(t,0) = u(t,l) = 0, &\quad t > 0, \\ (u, \partial_t u)(0,x) = (0, \sin\left(\frac{\pi}{l} x\right)), &\quad 0 < x < l. \end{aligned}$$

Solution. Setting $u(t,x) = T(t)X(t)$, then T and R satisfy

$$XT'' - TX'' = 0.$$

We denote λ such that

$$\frac{T''}{T} = \frac{X''}{X} = -\lambda.$$

For X, we have

$$X'' + \lambda X = 0,$$

solving the second order differential equation. If $\lambda < 0$, then the general solution is

$$X(x) = a_1 e^{\sqrt{-\lambda x}} + a_2 e^{-\sqrt{-\lambda x}},$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = a_2 = 0,$$

therefore

$$X(x) = 0.$$

If $\lambda = 0$, then the general solution is

$$X(x) = a_1 + a_2 x,$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = a_2 = 0,$$

therefore

$$X(x) = 0.$$

If $\lambda < 0$, then the general solution is

$$X(x) = a_1 \cos(\sqrt{\lambda} x) + a_2 \sin(\sqrt{\lambda} x),$$

by the boundary condition at $x = 0$ and $x = l$, we have

$$a_1 = 0, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2,$$

for all $n \in \mathbb{N}$. Therefore

$$X_n(x) = a_n \sin\left(\frac{n\pi}{l} x\right),$$

for all $n \in \mathbb{N}$, where a_n is to be determined.

For T, we have

$$T'' + \lambda T = 0,$$

solving the first order differential equation, the general solution is

$$T(t) = b_1 \cos(\sqrt{\lambda} t) + b_2 \sin(\sqrt{\lambda} t).$$

By the above discussion of X, we have

$$T_n(t) = b_{1,n} \cos\left(\frac{n\pi}{l} t\right) + b_{2,n} \sin\left(\frac{n\pi}{l} t\right),$$

for all $n \in \mathbb{N}$ where $b_{1,n}$ and $b_{2,n}$ are to be determined.

Therefore we seek the solution in the following form

$$u(t,x) = \sum_{n=1}^{\infty} \left(c_{1,n} \cos\left(\frac{n\pi}{l} t\right) + c_{2,n} \sin\left(\frac{n\pi}{l} t\right)\right) \sin\left(\frac{n\pi}{l} x\right).$$

By the initial condition, we have

$$c_{1,n} = 0,$$
$$c_{2,n} = \frac{2}{n\pi}\int_0^l \sin\left(\frac{\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)dx = \begin{cases} \frac{l}{\pi}, & n = 1, \\ 0, & n \geq 2, \end{cases}$$

then the solution is

$$u(t,x) = \frac{l}{\pi}\sin\left(\frac{\pi}{l}t\right)\sin\left(\frac{\pi}{l}x\right).$$

A Supplementary Problem

Problem 4. Show that the eigenvalue problem for second order differential equation

$$-\frac{d^2\omega}{dx^2}(x) - \lambda\omega(x) = 0, \quad 0 < x < l,$$
$$\omega(0) = \omega(l) = 0,$$

has the following properties.

(1) There exists an infinite number of real positive eigenvalues that can be arranged in increasing order $0 < \lambda_1 < \cdots < \lambda_n < \cdots$ that $\lim_{n\to\infty} \lambda_n = \infty$.
(2) The set of eigenfunctions $\{\omega_n\}_{n\geq 1}$ form an orthogonal basis in $L^2(0,l)$, i.e. for $n \neq m$,
$$\langle \omega_n, \omega_m \rangle := \int_0^l \omega_n(x)\omega_m(x)dx = 0,$$
and for arbitrary $f(x) \in L^2(0,l)$, there exists a set $\{c_n\}_{n\geq 1} \subset \mathbb{R}$ such that
$$\lim_{N\to\infty} \left\| f - \sum_{n=1}^N c_n\omega_n \right\|_{L^2(0,l)} = 0,$$
where
$$c_n = \frac{\int_0^l f(x)\omega_n(x)dx}{\int_0^l \omega_n(x)^2 dx},$$
for $n \in \mathbb{N}$.

For more materials, please refer to [1, 2, 3, 4].

References

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